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ANALYTIC SOLUTION OF THE EQUATIONS OF MOTION OF AN INTERPLANETARY SPACE VEHICLE IN THE MIDCOURSE PHASE OF ITS FLIGHT

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ABSTRACT

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The midcourse phase of a ballistic interplanetary flight is approximated by an elliptical reference trajectory. Linear perturbation theory is used to define the actual path of the vehicle relative to the reference path. The equations of motion for the actual path constitute a sixth-order linear system with variable coefficients. Two methods are presented for solving this system analytically to yield the variations in position and velocity as a continuous function of time.

The first solution exploits the fact that variations in the reference trajectory plane are uncoupled from variations normal to the plane. Thus, by the choice of a coordinate system in which one of the three orthogonal axes is perpendicular to the reference trajectory plane, the sixth-order system is separated into two independent systems, one of fourth order and one of second order. Both of these sub-systems are solved by direct integration.

The second method of solution utilizes the fact that the actual trajectory, like the reference trajectory, is an ellipse, but the orbital elements of the two ellipses are not identical. Linear theory is used to determine variations in position and velocity as a function of variations in the six orbital elements. It is shown that the six constants of integration obtained in the first solution can be expressed in terms of the variations of the orbital elements, and hence the two forms of solution are equivalent. AUTHOR — Stop

The six-component state vector at any time t is defined as the vector consisting of the three components of position variation and the three components of velocity variation at that time. The state vector at t_j is related to the state vector at t_i by means of a 6-by-6 matrix known as the transition matrix. Analytic expressions for the elements of the transition matrix are obtained for any arbitrary values of t_i and t_j.

It is assumed that a single application of impulsive thrust is to be used during the midcourse phase to correct the vehicle's path so that it reaches its destination at the proper time. In the linear theory the three components of

the required step change in velocity of the vehicle are related to the three components of the predicted position deviation at the destination by means of a 3-by-3 matrix known as the correction matrix. Analytic expressions for the elements of the correction matrix are derived. The uncoupling feature causes four of the nine elements of the matrix to be identically zero.

There is a brief discussion of the ways in which the equations that have been developed can be utilized on a manned interplanetary mission. Author

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NOMENCLATURE

General Notation

An asterisk over a capital letter indicates a matrix.

An underlined lower-case letter indicates a vector, which is equivalent to a one-column matrix. A vector symbol without the underlining indicates the magnitude of the vector.

A single dot over a symbol indicates the first derivative with respect to time of the quantity represented by the symbol. Two dots indicate the second derivative with respect to time. Time derivatives of vectors are taken with respect to an inertial coordinate system.

English Symbols

a	semi-major axis of reference trajectory
a_i	i-component of acceleration vector
$\overset{*}{A}_1, \ldots, \overset{*}{A}_4$	2-by-2 sub-matrices of $\overset{*}{Y}_4$ (See Section 8.)
$\underline{c}$	midcourse velocity correction vector
C	constant of integration (See Section 4.)
$\overset{*}{C}_{ji}$	transition matrix; 6-by-6 matrix relating components of $\delta \underline{x}_j$ to components of $\delta \underline{x}_i$
$\underline{d}_i$	position vector of space vehicle relative to i-th disturbing body (See Section 1.)
D	operator representing first derivative with respect to time (See Section 4.)
e	eccentricity of reference trajectory
$\underline{e}$	six-component vector consisting of variations in the orbital elements
E	eccentric anomaly on reference trajectory
E_M	one half the difference between E_j and E_i (See Section 8.)

E_P	one half the sum of E_j and E_i (See Section 8.)
f	true anomaly on reference trajectory (See Figure 2.)
f'	true anomaly on actual trajectory
$\dot{f}$	rate of change of true anomaly; angular velocity of r s z coordinate system
F	operator representing first derivative with respect to true anomaly (See Section 4.)
g	angle between velocity vector on reference trajectory and y-axis (See Figure 2.)
$\dot{g}$	angular velocity of p q z coordinate system
G	constant of gravitation
$\overset{*}{G}$	3-by-3 matrix relating components of $\delta \underline{\ddot{r}}$ to components of $\delta \underline{r}$ (See Section 1.)
h	orbital angular momentum per unit mass of space vehicle
i	inclination angle of reference trajectory plane
δi	angle between z'-axis and z-axis (See Figure 3.)
$\overset{*}{I}_N$	N-by-N identity matrix
$\overset{*}{J}_{ij}$	3-by-3 matrix relating components of $\delta \underline{v}_i$ to components of $\delta \underline{r}_i$ when $\delta \underline{r}_j$ is constant (See Section 9.)
$k_1, \ldots, k_6$	constants of integration (See Sections 4 & 5.)
$\overset{*}{K}_{CD}$	3-by-3 correction matrix
$\overset{*}{K}_{ij}$	3-by-3 matrix relating components of $\delta \underline{v}_i$ to components of $\delta \underline{r}_j$ when $\delta \underline{r}_i$ is constant (See Section 9.)
m	mass of space vehicle
m_o	mass of sun
m_i	mass of i-th disturbing body (See Section 1.)
M	mean anomaly on reference trajectory

M_o	value of mean anomaly at t = 0 (epoch)
$\overset{*}{M}_{ji}$	3-by-3 matrix relating components of $\delta \underline{r}_j$ to components of $\delta \underline{r}_i$ when $\delta \underline{v}_i$ is constant. (See Section 8.)
n	number of disturbing bodies (See Section 1.)
n	mean angular motion (See Section 6.)
$\overset{*}{N}_{ji}$	3-by-3 matrix relating components of $\delta \underline{r}_j$ to components of $\delta \underline{v}_i$ when $\delta \underline{r}_i$ is constant. (See Section 8.)
$\underline{O}$	zero vector
$\overset{*}{O}_N$	N-by-N zero matrix
p	distance along first axis of p q z coordinate system (See Figure 2.)
$\overset{*}{P}$	contribution of central body to $\overset{*}{G}$ (See Section 1.)
q	distance along second axis of p q z coordinate system (See Figure 2.)
$\overset{*}{Q}$	contribution of disturbing bodies to $\overset{*}{G}$ (See Section 1.)
r	distance along first axes of r s z coordinate system (See Figure 2.)
$\underline{r}$	position vector of space vehicle on reference trajectory
$\underline{r}'$	position vector of space vehicle on actual trajectory
$\dot{\underline{r}}$	velocity vector of space vehicle on reference trajectory
$\underline{r}_i$	position vector of i-th disturbing body (See Section 1.)
R	disturbing function (See Section 1.)
RTP	reference trajectory plane
s	distance along second axis of r s z coordinate system (See Figure 2.)
$\overset{*}{S}_{ji}$	3-by-3 matrix relating components of $\delta \underline{v}_j$ to components of $\delta \underline{r}_i$ when $\delta \underline{v}_i$ is constant (See Section 8.)
t	time

t_o	time of perihelion passage for reference trajectory
$\overset{*}{T}_{ji}$	3-by-3 matrix relating components of $\delta \underline{v}_j$ to components of $\delta \underline{v}_i$ when $\delta \underline{r}_i$ is constant (See Section 8.)
u	integration variable (See Section 4.)
v	integration variable (See Section 4.)
v	magnitude of velocity vector (See Section 7.)
v_i	i-component of velocity vector
w	integration variable (See Section 4.)
x	distance along first axis of non-rotating x y z coordinate system (See Figure 2.)
$\underline{x}$	six-component vector consisting of $\delta \underline{r}$ and $\delta \underline{v}$
X	singularity factor (See Section 9.)
y	distance along second axis of non-rotating x y z coordinate system (See Figure 2.)
$\overset{*}{Y}$	6-by-6 matrix relating components of $\underline{x}$ to components of $\underline{e}$ (See Section 8.)
$\overset{*}{Y}_4$	4-by-4 submatrix of $\overset{*}{Y}$
z	distance normal to reference trajectory plane

Greek Symbols

γ	flight path angle on reference trajectory (See Figure 2.)
δ	operator signifying first variation
∇	gradient of a scalar quantity
μ	gravitational invariant in sun's gravitational field
$\sum$	summation symbol
ϕ	longitude of perihelion of reference trajectory
$\delta\phi$	longitude of perihelion of actual trajectory relative to perihelion of reference trajectory
ω	latitude of perihelion of reference trajectory

$\delta\omega$	angle, in actual trajectory plane, between positive half of line of nodes and x'-axis. (See Figure 3.)
Ω	longitude of ascending node of reference trajectory
$\delta\Omega$	angle, in reference trajectory plane, between x-axis and positive half of line of nodes (See Figure 3.)

Superscripts

T	transpose of a vector or matrix
-1	inverse of a square matrix
'	pertaining to actual trajectory as opposed to reference trajectory (See Section 6.)
-	pertaining to variant path before application of midcourse correction
+	pertaining to variation path after application of midcourse correction

Subscripts

C	corresponding to time of midcourse velocity correction
D	corresponding to nominal time of arrival at destination
i	general index; $i = 1, \ldots, n$
i	corresponding to time t_i
j	corresponding to time t_j
p	component along p-axis
q	component along q-axis
r	component along r-axis
s	component along s-axis
x	component along x-axis
y	component along y-axis
z	component along z-axis

1. Introduction

The vector equation of motion of a space vehicle in a gravitational field is

$$\ddot{\underline{r}} + \frac{\mu}{r^3}\underline{r} = \nabla R \tag{1-1}$$

For the midcourse section of interplanetary flights $\underline{r}$ is normally defined as the position vector of the vehicle relative to the center of the sun. The vehicle's acceleration vector is $\ddot{\underline{r}}$. The symbol r, without the underlining, represents the magnitude of $\underline{r}$.

The gravitational parameter μ is defined by

$$\mu = G\ (m_o + m) \tag{1-2}$$

where G is the constant of gravitation, and m_o and m are, respectively, the mass of the sun and the mass of the space vehicle. For all practical purposes,

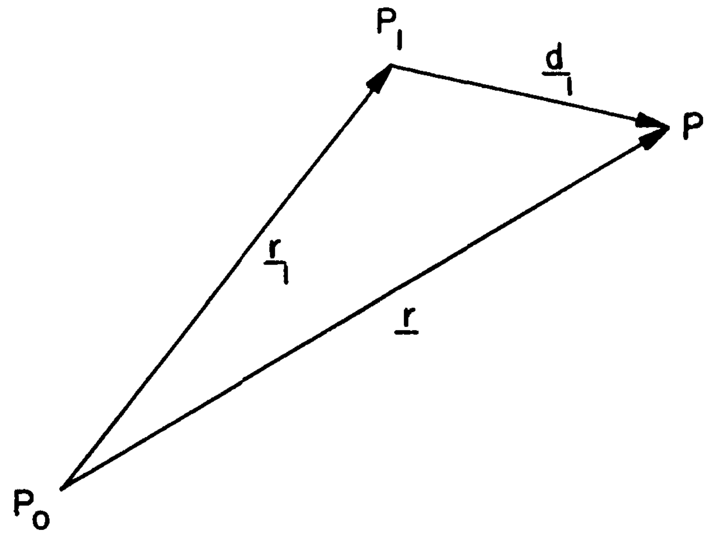
$$\mu = Gm_o \tag{1-3}$$

The vector quantity ∇R is the gradient of the disturbing function R.

$$R = G \sum_{i=1}^{n} m_i \left(\frac{1}{d_i} - \frac{\underline{r} \cdot \underline{r}_i}{r_i^{\ 3}} \right) \tag{1-4}$$

∇R represents the effect on vehicle motion caused by the gravitational fields of the "disturbing" masses, usually planets. d_i is the distance from the vehicle of the i-th planet, whose mass is m_i. $\underline{r}_i$ is the position vector of the i-th planet with respect to the sun's center. There are n disturbing masses.

The position relationships of the space vehicle relative to the sun and one disturbing planet are indicated in Figure 1.



P, P_o, P_1 – three bodies treated as hypothetical point-masses

P – space vehicle

P_o – sun

P_1 – "disturbing" planet

Figure 1. Vector Diagram for the Three-Body Problem

In the formulation of Equation (1-1), the space vehicle, sun, and disturbing bodies are all assumed to be point masses. A more exact equation is obtained if oblateness effects are taken into account.

A nominal, or reference, trajectory for an interplanetary voyage is generated by numerical integration of Equation (1-1), modified to include the effects of Earth's oblateness. The initial conditions for the integration are carefully chosen so that a suitable set of end conditions (position and velocity relative to the destination planet) is achieved.

The actual trajectory of a space vehicle differs from the pre-computed reference trajectory due to imperfect instrumentation and inexact guidance equations. Because the differences between the two trajectories are assumed to be small, linear perturbation theory is used to express the equations of motion in simple matrix form. The dependent variable $\underline{r}$ in Equation (1-1) is replaced by $\delta\underline{r}$, the variation in position between actual trajectory and reference trajectory. The matrix equation is

$$\delta\ddot{\underline{r}} = \overset{*}{G}\ \delta\underline{r} \tag{1-5}$$

An asterisk over a capital letter signifies a matrix. The matrix $\overset{*}{G}$ may be subdivided into two matrices, one giving the effect of the central body and the other giving the effect of the disturbing bodies.

$$\overset{*}{G} = \overset{*}{P} + \overset{*}{Q} \tag{1-6}$$

where

$$\overset{*}{P} = \frac{\mu}{r^3}\left(\frac{3\ \underline{r}\ \underline{r}^T}{r^2} - \overset{*}{I}_3\right) \tag{1-7}$$

$$\overset{*}{Q} = G\sum_{i=1}^{n}\frac{m_i}{d_i^{\,3}}\left(\frac{3\ \underline{d}_i\ \underline{d}_i^{\,T}}{d_i^{\,2}} - \overset{*}{I}_3\right) \tag{1-8}$$

The superscript T refers to the transpose of a vector or matrix. $\overset{*}{I}_3$ is the 3-by-3 identity matrix. $\overset{*}{G}$, $\overset{*}{P}$, and $\overset{*}{Q}$ are all symmetric 3-by-3 matrices.

The derivation of Equations (1-5) through (1-8) is given in Reference (1).

In the midcourse phase of an interplanetary flight, when the space vehicle is appreciably beyond the sphere of influence of the launch planet and is still a considerable distance from the sphere of influence of the destination planet, the motion of the vehicle is essentially two-body motion in the sun's gravitational field. Under these conditions Equations (1-1) and (1-5) become, respectively,

$$\ddot{\underline{r}} + \frac{\mu}{r^3}\underline{r} = \underline{0} \tag{1-9}$$

$$\delta\ddot{\underline{r}} = \overset{*}{P}\ \delta\underline{r} \tag{1-10}$$

The analytic solution of (1-9) is the familiar equation of the conic section. It is the objective of this paper to find and exploit an analytic solution of (1-10).

It should be noted that the reference trajectory is obtained by numerical integration of a modified form of (1-1); it is not a true conic section. The variations used in (1-10) are the variations from the reference trajectory, not variations from a conic section. Thus, the disturbing effects of the planets are taken into account in determining the basic reference trajectory but are neglected in determining variations in acceleration due to variations in position.

2. Coordinate Systems

Because conic sections, which constitute the family of solutions of (1-9), are planar curves, it is expedient to choose a three-dimensional coordinate system in which two of the three axes lie in the plane of the conic section solution and the third axis is normal to that plane. The plane will be referred to as the "reference trajectory plane" (RTP).

Three different coordinate systems have been found useful in the analysis of interplanetary midcourse motion. All three are right-handed rectilinear systems in which the origin is at the center of the sun and the z-axis is perpendicular to the RTP, positive in the direction of the vehicle's orbital angular momentum vector.

In the x y z system the axes are non-rotating. The x-axis lies in the direction of perihelion; the y-axis is in the direction of the latus rectum. This system is most appropriate for computing the reference trajectory from Equation (1-1) or when Equation (1-5) must be used for the variational analysis.

The r s z system rotates about the z-axis with angular velocity $\dot{f}$, where f is the true anomaly on the reference trajectory. The r-axis is in the direction of the position vector $\underline{r}$; the s-axis is in the transverse direction, 90° ahead of the r-axis in the direction of vehicle motion. The r s z system is used in obtaining the analytic solution of (1-10).

The third coordinate system is the p q z system, which rotates about the z-axis with angular velocity $\dot{g}$, where g is the angle between the velocity vector $\dot{\underline{r}}$ and the y-axis. The q-axis is in the direction of $\dot{\underline{r}}$; the p-axis is 90° behind the q-axis. This system is the most convenient one for expressing the analytic forms of the elements of the basic guidance matrices that are defined in the later sections of this paper.

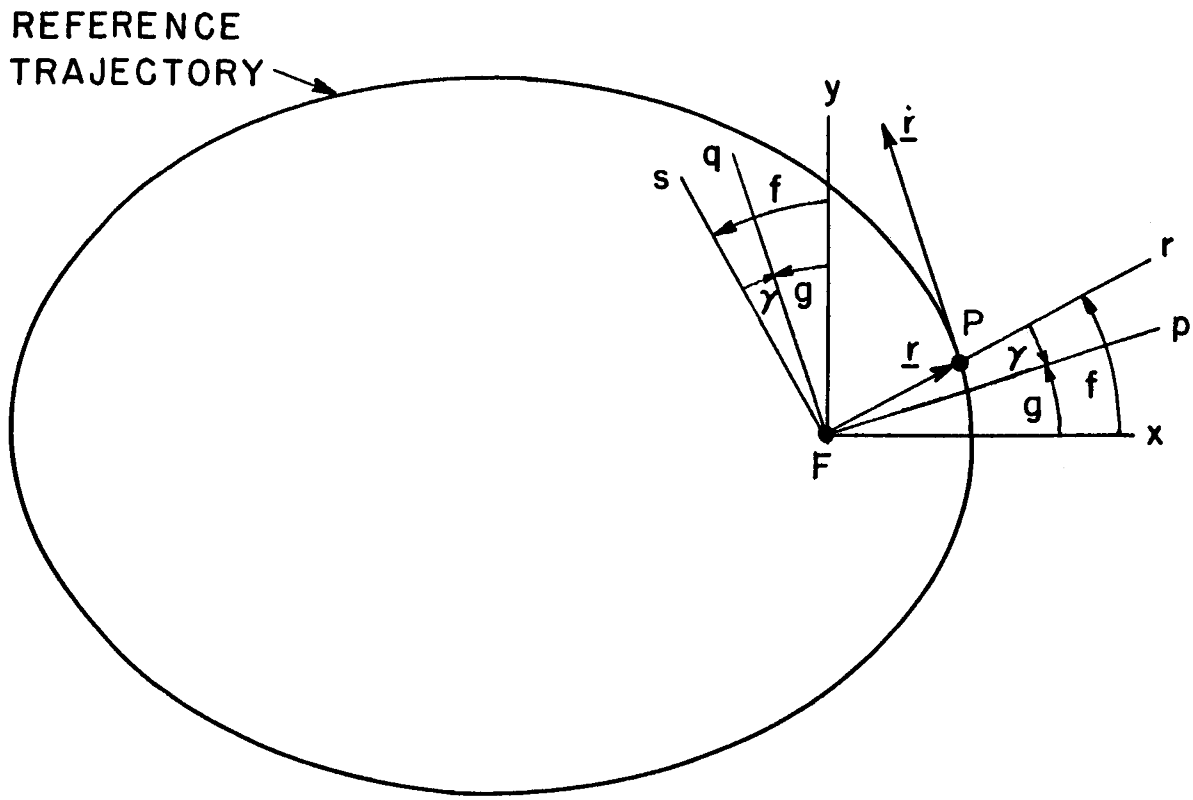
The angular relations among the three systems are shown in Figure 2.

3. Equations of Motion in Component Form

In the x y z coordinate system the component equations of (1-9) are

$$\begin{bmatrix} \ddot{x} \\ \\ \ddot{y} \end{bmatrix} + \frac{\mu}{r^3} \begin{bmatrix} x \\ \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} \qquad (3\text{-}1)$$

No z-axis equation is needed since z is identically zero for all values of t.



Fq is parallel to $\underline{v}$.

F – attractive focus (center of sun)

P – vehicle position on reference trajectory

$\underline{r}$ – position vector

$\dot{\underline{r}}$ – velocity vector

f = $\sphericalangle$ xFr = $\sphericalangle$ yFs = true anomaly = angle between r s z coordinate system and x y z coordinate system

γ = $\sphericalangle$ rFp = $\sphericalangle$ sFq = flight path angle = angle between p q z coordinate system and r s z coordinate system

g = $\sphericalangle$ xFp = $\sphericalangle$ yFq = f − γ

= angle between p q z coordinate system and x y z coordinate system

Figure 2. Orientations of Reference Trajectory Coordinate Systems

In the r s z coordinate system, both s and z are identically zero. The component equations are

$$\begin{bmatrix} \ddot{r} - r\dot{f}^2 \\ \\ r\ddot{f} + 2\dot{r}\dot{f} \end{bmatrix} + \frac{\mu}{r^3}\begin{bmatrix} r \\ \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} \tag{3-2}$$

These equations can be integrated directly, as shown in any basic text on celestial mechanics such as Reference (2). The solution is

$$r = \frac{h^2/\mu}{1 + e \cos f} \tag{3-3}$$

where

$$h = r^2 \dot{f} \tag{3-4}$$

The constant h is the orbital angular momentum per unit mass of the space vehicle. e is the eccentricity of the conic section.

When the r s z system is used to expand Equation (1-10), $\overset{*}{P}$ becomes a diagonal matrix.

$$\overset{*}{P} = \frac{\mu}{r^3}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{3-5}$$

The variations in position, velocity, and acceleration are given by

$$\delta \underline{r} = \begin{bmatrix} \delta r \\ \delta s \\ \delta z \end{bmatrix} = \begin{bmatrix} \delta r \\ r \delta f \\ \delta z \end{bmatrix} \tag{3-6}$$

$$\delta \dot{\underline{r}} = \begin{bmatrix} \delta \dot{r} - r \dot{f} \delta f \\ \dot{f} \delta r + r \delta \dot{f} + \dot{r} \delta f \\ \delta \dot{z} \end{bmatrix} \tag{3-7}$$

$$\delta \ddot{\underline{r}} = \begin{bmatrix} \delta \ddot{r} - \dot{f}^2 \delta r - 2 r \dot{f} \delta \dot{f} - (r \ddot{f} + 2 \dot{r} \dot{f}) \delta f \\ 2 \dot{f} \delta \dot{r} + \ddot{f} \delta r + r \delta \ddot{f} + 2 \dot{r} \delta \dot{f} + (\ddot{r} - r \dot{f}^2) \delta f \\ \delta \ddot{z} \end{bmatrix} \tag{3-8}$$

When Equations (3-2), (3-5), and (3-8) are substituted into (1-10), the resulting matrix equation is

$$\begin{bmatrix} \delta \ddot{r} - \dot{f}^2 \delta r - 2 r \dot{f} \delta \dot{f} \\ 2 \dot{f} \delta \dot{r} + \ddot{f} \delta r + r \delta \ddot{f} + 2 \dot{r} \delta \dot{f} \\ \delta \ddot{z} \end{bmatrix} = \frac{\mu}{r^3} \begin{bmatrix} 2 \delta r \\ 0 \\ - \delta z \end{bmatrix} \tag{3-9}$$

The equations of (3-9) constitute the variant equations of motion in the r s z coordinate system. There are three second-order linear differential equations with time-varying coefficients. The first two equations are coupled equations in the dependent variables δr and δf. The third equation contains only one dependent variable, δz. Consequently, the system, which is basically of sixth order, can be separated into two uncoupled subsystems, one of fourth order and the other of second order. In the following sections each of the two subsystems is integrated analytically.

4. Variant Motion in the Reference Trajectory Plane

If the operator D is used to signify the first derivative with respect to time, the first two equations of (3-9) may be written as

$$\begin{bmatrix} D^2 - \dot{f}^2 - \frac{2\mu}{r^3} & -2r\dot{f}D \\ 2\dot{f}D + \ddot{f} & (rD + 2\dot{r})D \end{bmatrix} \begin{bmatrix} \delta r \\ \delta f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4-1}$$

It is noteworthy that the equations depend on the first and second time derivatives of δf, but not on δf itself. Then it can be seen by inspection that one solution of the coupled equations of (4-1) is

$$\delta r = 0 \qquad \delta f = k_1 \tag{4-2}$$

where k_1 is an arbitrary constant.

The system represented by (4-1) may be regarded as a third-order system in the dependent variables δr and $\delta \dot{f}$. It is simplified if the independent variable is changed from t to f. Let operator F represent the first derivative with respect to f. Then,

$$D = \dot{f} F \tag{4-3}$$

$$D^2 = \dot{f} F (\dot{f} F) = \dot{f}^2 \left(F - \frac{2e \sin f}{1 + e \cos f} \right) F \tag{4-4}$$

When these relations are used in conjunction with (3-3) and (3-4), the differential equations of (4-1) become

$$\begin{bmatrix} (1 + e\cos f)F^2 - (2e\sin f)F - (3 + e\cos f) & -2 \\ & \\ 2\left[(1 + e\cos f)F - e\sin f\right] & F \end{bmatrix} \begin{bmatrix} \delta r \\ \\ \frac{h^2}{\mu} F\, \delta f \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} \tag{4-5}$$

The variable δf may be eliminated from the two equations to yield a single equation in δr.

$$\left[(1 + e\cos f)F^3 - (3e\sin f)F^2 + (1 + e\cos f)F - (3e\sin f)\right]\delta r = 0 \tag{4-6}$$

The terms of this equation may be re-arranged as follows:

$$\left[(1 + e\cos f)F - (3e\sin f)\right](F^2 + 1)\ \delta r = 0 \tag{4-7}$$

Two solutions of (4-7) are obtained from

$$(F^2 + 1)\,\delta r = 0 \tag{4-8}$$

These solutions are obviously $k_2 \cos f$ and $k_3 \sin f$. For the former,

$$\frac{h^2}{\mu} F\,(\delta f) = -k_2\left[2\cos f + e\,(\cos^2 f - \sin^2 f)\right] \tag{4-9}$$

Upon integration,

$$\delta f = -k_2 \frac{\mu}{h^2}\,(2 + e\cos f)\sin f \tag{4-10}$$

Similarly, for the solution $k_3 \sin f$,

$$\frac{h^2}{\mu} F(\delta f) = -2k_3 (1 + e \cos f) \sin f \tag{4-11}$$

$$\delta f = k_3 \frac{\mu}{h^2} (2 + e \cos f) \cos f \tag{4-12}$$

The fourth solution for the variant motion in the RTP can be obtained by the method of variation of parameters, which is described in the first chapter of Reference (3). A new variable w is introduced and is defined by

$$w = (F^2 + 1)\ \delta r \tag{4-13}$$

In terms of this new variable, Equation (4-7) becomes

$$\frac{dw}{df} - \frac{3 e \sin f}{1+e \cos f} w = 0 \tag{4-14}$$

The variables w and f are separable.

$$\frac{dw}{w} - \frac{3 e \sin f}{1+e \cos f} df = 0 \tag{4-15}$$

The result of integrating this equation is

$$\log w + 3 \log (1 + e \cos f) = \log C \tag{4-16}$$

where C is an arbitrary constant. Then,

$$w = (F^2 + 1)\,\delta r = \frac{C}{(1 + e\cos f)^3} \tag{4-17}$$

Since the homogeneous solutions of (4-17) for δr are $\cos f$ and $\sin f$, it is assumed that

$$\delta r = u\cos f + v\sin f \tag{4-18}$$

where u and v are functions of f which satisfy the following criteria:

$$\cos f \frac{du}{df} + \sin f \frac{dv}{df} = 0 \tag{4-19}$$

$$-\sin f \frac{du}{df} + \cos f \frac{dv}{df} = \frac{C}{(1 + e\cos f)^3} \tag{4-20}$$

The solution for $\frac{du}{df}$ and $\frac{dv}{df}$ is

$$\frac{du}{df} = -\frac{C\sin f}{(1 + e\cos f)^3} \tag{4-21}$$

$$\frac{dv}{df} = \frac{C\cos f}{(1 + e\cos f)^3} \tag{4-22}$$

The first of these two equations may be integrated directly.

$$du = \frac{C}{e}\,\frac{d\,(1 + e\cos f)}{(1 + e\cos f)^3} \tag{4-23}$$

$$u = -\frac{C}{2e(1 + e\cos f)^2} \qquad (4\text{-}24)$$

Integration of Equation (4-22) is less obvious. It can be handled by making a change in the independent variable; the change to be made depends on the nature of the conic section that constitutes the reference trajectory. In this paper, which is concerned with midcourse characteristics, the reference trajectory is assumed to be an ellipse. The new variable is the eccentric anomaly E, which can be related to f by rewriting Equation (3-3) as follows:

$$r = \frac{a(1 - e^2)}{1 + e\cos f} = a(1 - e\cos E) \qquad (4\text{-}25)$$

where a is the semi-major axis of the reference trajectory. The quantity $a(1 - e^2)$ is equal to the length of the semi-latus rectum of the reference trajectory.

$$\frac{h^2}{\mu} = a(1 - e^2) \qquad (4\text{-}26)$$

The transformations from f to E, and vice versa, are accomplished by the following equations:

$$\sin f = \frac{(1 - e^2)^{1/2}\sin E}{1 - e\cos E} \qquad \sin E = \frac{(1 - e^2)^{1/2}\sin f}{1 + e\cos f} \qquad (4\text{-}27)$$

$$\cos f = \frac{\cos E - e}{1 - e\cos E} \qquad \cos E = \frac{\cos f + e}{1 + e\cos f} \qquad (4\text{-}28)$$

$$df = \frac{(1-e^2)^{1/2}\, dE}{1-e \cos E} \qquad dE = \frac{(1-e^2)^{1/2}\, df}{1+e \cos f} \tag{4-29}$$

$$(1 + e \cos f)\,(1 - e \cos E) = 1 - e^2 \tag{4-30}$$

Equation (4-22) becomes

$$dv = \frac{C \cos f}{(1+e \cos f)^3}\, df$$

$$= C\, \frac{(\cos E - e)}{1 - e \cos E}\, \frac{(1-e \cos E)^3}{(1-e^2)^3}\, \frac{(1-e^2)^{1/2}\, dE}{1-e \cos E}$$

$$= \frac{C}{(1-e^2)^{5/2}} \left[- e + (1+e^2) \cos E - e \cos^2 E \right] dE \tag{4-31}$$

The integral of this equation is

$$v = \frac{C}{2(1-e^2)^{5/2}} \left\{ - 3 e E + \left[2 (1+e^2) - e \cos E \right] \sin E \right\} \tag{4-32}$$

By means of Kepler's equation

$$M = E - e \sin E \tag{4-33}$$

the mean anomaly M may be introduced to replace E in the secular term of (4-32), and the equation can then be written in terms of the time-varying quantities M and f.

$$v = \frac{C}{2(1-e^2)^{5/2}} \left[-3eM + (2 - e^2 - e\cos E)\sin E \right]$$

$$= \frac{C}{2(1-e^2)^{5/2}} \left\{ -3eM + \left[2 - e^2 - \frac{e(\cos f + e)}{1 + e\cos f} \right] \frac{(1-e^2)^{1/2}\sin f}{1 + e\cos f} \right\}$$

$$= \frac{C}{2(1-e^2)} \left[-\frac{3eM}{(1-e^2)^{3/2}} + \frac{(2 + e\cos f)\sin f}{(1 + e\cos f)^2} \right] \qquad (4\text{-}34)$$

No additional constants of integration are needed in (4-24) and (4-34), because such constants would simply be multiplied by $\cos f$ and $\sin f$, respectively, in the determination of δr; hence they can be incorporated into k_2 and k_3.

Equations (4-24) and (4-34) are substituted into (4-18).

$$\delta r = C \left[-\frac{\cos f}{2e(1 + e\cos f)^2} - \frac{3Me\sin f}{2(1-e^2)^{5/2}} + \frac{(2 + e\cos f)\sin^2 f}{2(1-e^2)(1 + e\cos f)^2} \right]$$

$$= \frac{C}{1-e^2} \left[-\frac{3Me\sin f}{2(1-e^2)^{3/2}} + \frac{1}{1 + e\cos f} - \frac{\cos f}{2e} \right] \qquad (4\text{-}35)$$

The constant $-\dfrac{C}{2e(1-e^2)}$ may be incorporated into k_2, so that the last term in (4-35) is eliminated. Finally, the fourth solution for δr is

$$\delta r = k_4 \left[-\frac{3 M e \sin f}{2(1-e^2)^{3/2}} + \frac{1}{1 + e \cos f} \right] \tag{4-36}$$

where

$$k_4 = \frac{C}{1 - e^2} \tag{4-37}$$

The fourth solution for δf is obtained from the upper equation of (4-5). The first and second derivatives of δr with respect to f must first be determined. By utilizing the fact that

$$F(M) = \frac{(1-e^2)^{3/2}}{(1 + e \cos f)^2} \tag{4-38}$$

it is found that

$$F(\delta r) = -\frac{k_4 e}{2} \left[\frac{3 M \cos f}{(1-e^2)^{3/2}} + \frac{\sin f}{(1 + e \cos f)^2} \right] \tag{4-39}$$

$$F^2(\delta r) = k_4 \left[\frac{3 M e \sin f}{2(1-e^2)^{3/2}} - \frac{1}{1 + e \cos f} + \frac{1-e^2}{(1 + e \cos f)^3} \right] \tag{4-40}$$

Then, from (4-5),

$$F(\delta f) = \frac{3 k_4 \mu}{2 h^2} \left[\frac{2 M (1 + e \cos f) e \sin f}{(1 - e^2)^{3/2}} - 1 \right] \tag{4-41}$$

This equation is integrated to solve for δf.

$$d(\delta f) = \frac{3 k_4 \mu}{2 h^2} \left[- \frac{2 M (1 + e \cos f) \, d (1 + e \cos f)}{(1 - e^2)^{3/2}} - df \right]$$

$$= \frac{3 k_4 \mu}{2 h^2} \left\{ - \frac{d \left[M (1 + e \cos f)^2 \right]}{(1 - e^2)^{3/2}} + \frac{(1 + e \cos f)^2 \, dM}{(1 - e^2)^{3/2}} - df \right\}$$

$$= - \frac{3 k_4 \mu}{2 h^2 (1 - e^2)^{3/2}} \, d \left[M (1 + e \cos f)^2 \right] \tag{4-42}$$

$$\delta f = - \frac{3 k_4 \mu}{2 h^2 (1 - e^2)^{3/2}} M (1 + e \cos f)^2 \tag{4-43}$$

No added constant of integration is needed in the last integration because of the presence of k_1.

The complete equations for δr and δs are

$$\delta r = k_2 \cos f + k_3 \sin f$$

$$+ k_4 \left(- \frac{3 M e \sin f}{2 (1 - e^2)^{3/2}} + \frac{1}{1 + e \cos f} \right) \tag{4-44}$$

$$\delta s = r\delta f = \frac{k_1 h^2}{\mu(1+e\cos f)} - \frac{k_2\,(2+e\cos f)\sin f}{1+e\cos f}$$

$$+ \frac{k_3\,(2+e\cos f)\cos f}{1+e\cos f} - \frac{3k_4\,M\,(1+e\cos f)}{2\,(1-e^2)^{3/2}} \tag{4-45}$$

5. Variant Motion Normal to the Reference Trajectory Plane

The equation for the variant motion along the z-axis is

$$\delta\ddot{z} + \frac{\mu}{r^3}\,\delta z = 0 \tag{5-1}$$

Comparison of this equation with the equations of (3-1) indicates that x and y are independent solutions for δz.

$$\delta z = k_5\,x + k_6\,y \tag{5-2}$$

$$= r\,(k_5 \cos f + k_6 \sin f) \tag{5-3}$$

$$= (k_5^{\,2} + k_6^{\,2})^{1/2}\, r \sin\left(f + \tan^{-1}\frac{k_5}{k_6}\right) \tag{5-4}$$

$$= \frac{h^2}{\mu\,(1+e\cos f)}\,(k_5 \cos f + k_6 \sin f) \tag{5-5}$$

6. Variations in the Orbital Elements

The last three sections have presented a method of determining the difference in position between the actual trajectory and the reference trajectory by formulating and then integrating the linearized differential equations of the variant motion. A second method is presented in this section. The fundamental premise of the second solution is that the actual trajectory, like the reference trajectory, is a conic section and that the two trajectories lie close to each other in space. The procedure involves the determination of the manner in which variations in each of the six orbital elements that characterize the vehicle's reference trajectory affect the vehicle's position as a function of time.

A common grouping of the orbital elements consists of the semi-major axis length a, the eccentricity e, the longitude Ω of the ascending node, the inclination i, the latitude ω of perihelion, and the time t_o of perihelion passage. In the following analysis the longitude ϕ of perihelion and the mean anomaly M_o at epoch are also used. Since only six elements are independent, the last two are linearly related to the first six.

$$\phi = \Omega + \omega \tag{6-1}$$

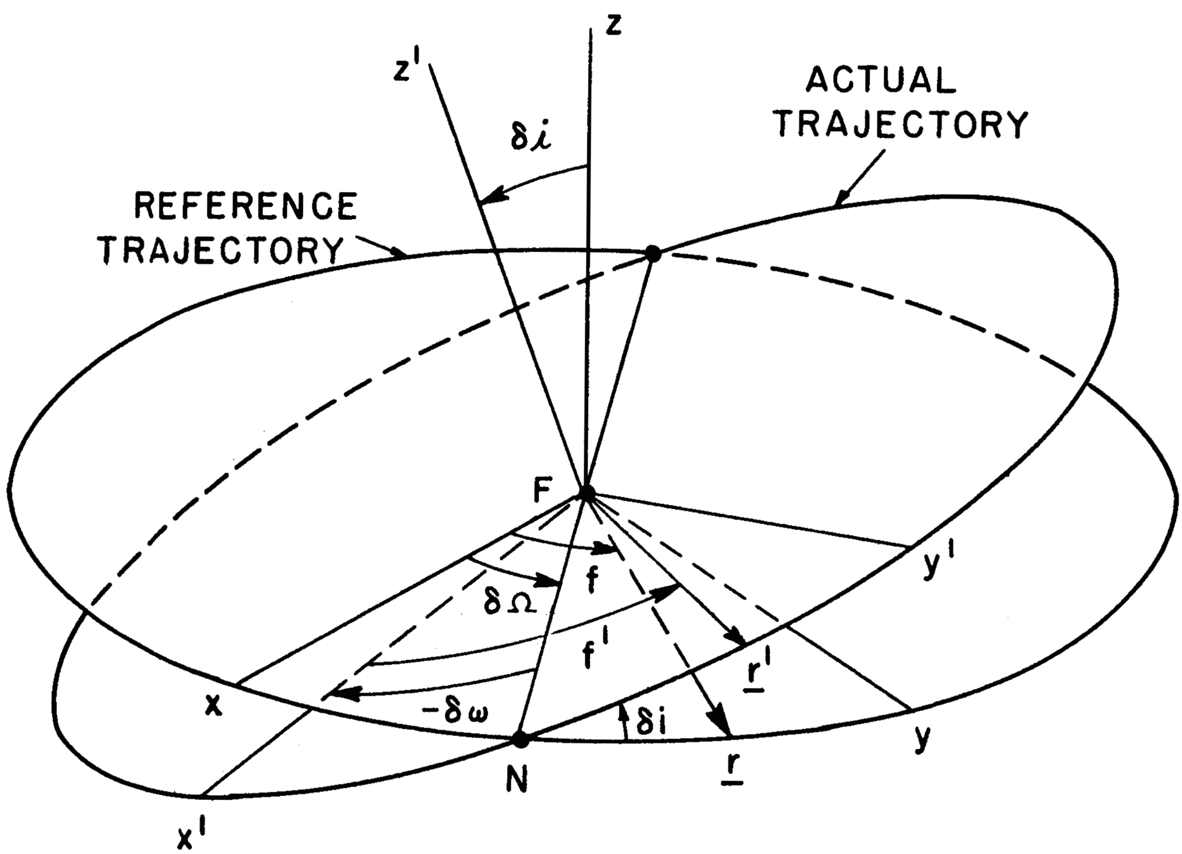
$$M_o = - n t_o \tag{6-2}$$

where n is the mean angular motion, i.e., the average angular velocity of the space vehicle in its elliptical orbit about the sun.

The analysis that follows is applicable to elliptical reference trajectories of moderate eccentricity. The word "moderate" is intended to signify that the eccentricity lies between zero and unity but is significantly different from either boundary value.

The space orientation of the actual trajectory relative to the reference trajectory is determined by the three angles $\delta\Omega$, $\delta\omega$, and δi. $\delta\Omega$ lies in the plane of the reference trajectory, $\delta\omega$ lies in the plane of the actual trajectory, and δi lies in the plane perpendicular to the line of nodes. These angles are illustrated in Figure 3. If all three angles are zero, the major and minor axes of the two trajectories coincide.

The constraint on the angular variations is that they be such that the difference in position between a point on the actual trajectory and a point on the



F	–	focus at center of sun
FN	–	line of nodes
Fx,Fy,Fz	–	axes of reference trajectory coordinate system
Fx',Fy',Fz'	–	axes of actual trajectory coordinate system
$\delta\Omega, \delta i, \delta\omega$	–	orientation angles between two coordinate systems
$\underline{r}$	–	position vector on reference trajectory at time t
$\underline{r}'$	–	position vector on actual trajectory at time t
f	–	true anomaly on reference trajectory at time t
f'	–	true anomaly on actual trajectory at time t

Figure 3. Orientation of Actual Trajectory Relative to Reference Trajectory

reference trajectory at any specified time be small. The constraint can be met even though $\delta\Omega$ and $\delta\omega$ are large angles, as long as their sum, which is equal to $\delta\phi$, is a small angle. The angle δi is always small. The variations in the anomalies (δf, δE, δM, δM_o) are also small.

The prime notation is used to distinguish quantities on the actual trajectory from the corresponding quantities on the reference trajectory. Thus, $\underline{r}'$ is the radius vector on the actual trajectory corresponding to $\underline{r}$ on the reference trajectory. The components of $\underline{r}'$ in the r s z coordinate system are designated r'_r, r'_s, and r'_z; the corresponding components of $\underline{r}$ are r, 0, and 0. From Figure 3, the components of position variation are

$$\begin{aligned}\delta r &= r'_r - r \\ &= r' \left[\cos(f' + \delta\omega)\cos(f - \delta\Omega)\right. \\ &\qquad \left. + \sin(f' + \delta\omega)\cos\delta i \sin(f - \delta\Omega)\right] - r \\ &= r' \cos(\delta f + \delta\phi) - r \\ &= r' - r\end{aligned} \tag{6-3}$$

$$\begin{aligned}\delta s &= r'_s - 0 \\ &= r' \left[-\cos(f' + \delta\omega)\sin(f - \delta\Omega)\right. \\ &\qquad \left. + \sin(f' + \delta\omega)\cos\delta i \cos(f - \delta\Omega)\right] \\ &= r' \sin(\delta f + \delta\phi) \\ &= r(\delta f + \delta\phi)\end{aligned} \tag{6-4}$$

$$\begin{aligned}\delta z &= r'_z - 0 \\ &= r' \sin(f' + \delta\omega)\sin\delta i \\ &= r\,\delta i \sin(f + \delta\omega) \\ &= r\,\delta i \sin(f - \delta\Omega)\end{aligned} \tag{6-5}$$

By the use of Equation (6-5) δz can be determined as a function of r and f, which are known time-varying functions of the reference trajectory, and the two orbital element variations δi and $\delta \Omega$. In Equations (6-3) and (6-4) the relations for the components δr and δs must be expressed in similar fashion in terms of the other orbital element variations. The procedure to be followed for δr is to write δr as a function of the orbital element variations and δE, then to find δE in terms of the orbital element variations, and finally to combine the two relations. In the case of δs, the first step involves expressing δf as a function of the variations in the elements and δE, then proceeding in a manner analogous to that for δr.

δr is obtained from the basic equation

$$r = a\,(1 - e \cos E) \tag{6-6}$$

The variational form obtained from this equation is

$$\delta r = a \left[(1 - e \cos E) \frac{\delta a}{a} - \cos E \; \delta e + e \sin E \; \delta E \right] \tag{6-7}$$

The relationship used to obtain δf is

$$(1 + e \cos f)(1 - e \cos E) = 1 - e^2 \tag{6-8}$$

The variational form is

$$\begin{aligned} &(1 + e \cos f)(- \cos E \; \delta e + e \sin E \; \delta E) \\ &+ (1 - e \cos E)(\cos f \; \delta e - e \sin f \; \delta f) = - 2 e \; \delta e \end{aligned} \tag{6-9}$$

This equation is solved for δf.

$$\delta f = \frac{\sin f}{1 - e^2} \; \delta e + \frac{1 + e \cos f}{(1 - e^2)^{1/2}} \; \delta E \tag{6-10}$$

Two expressions for the mean anomaly are used to determine δE.

$$M = n\,(t - t_o) = E - e \sin E \tag{6-11}$$

Then,

$$\delta M = (t - t_o)\, \delta n - n\, \delta t_o$$
$$= - \sin E\, \delta e + (1 - e \cos E)\, \delta E \qquad (6\text{-}12)$$

The solution for δE is

$$\delta E = \frac{1}{1 - e \cos E} \left(M \frac{\delta n}{n} - n\, \delta t_o + \sin E\, \delta e \right) \qquad (6\text{-}13)$$

$\frac{\delta n}{n}$ is not one of the orbital element variations being used; it can be expressed in terms of $\frac{\delta a}{a}$ by means of Kepler's third law of planetary motion, which states that

$$\mu = n^2 a^3 \qquad (6\text{-}14)$$

Since μ is invariant,

$$\delta \mu = 0 = 2 n a^3\, \delta n + 3 n^2 a^2\, \delta a \qquad (6\text{-}15)$$

and

$$\frac{\delta n}{n} = - \frac{3}{2} \frac{\delta a}{a} \qquad (6\text{-}16)$$

Equations (6-7), (6-13), and (6-16) are combined to express δr in terms of the variations in the orbital elements. All sinusoidal terms are written in terms of the true anomaly f rather than the eccentric anomaly E. δs is obtained from Equations (6-4), (6-10), (6-13), and (6-16), δz from Equation (6-5). The final equations for the components of $\delta \underline{r}$ are

$$\delta r = a \left[\left(\frac{1 - e^2}{1 + e \cos f} - \frac{3 M e \sin f}{2 (1 - e^2)^{1/2}} \right) \frac{\delta a}{a} \right.$$

$$\left. - \frac{e \sin f}{(1 - e^2)^{1/2}}\, n\, \delta t_o - \cos f\, \delta e \right] \qquad (6\text{-}17)$$

$$\delta s = a\left[-\frac{3M(1+e\cos f)}{2(1-e^2)^{1/2}}\frac{\delta a}{a} - \frac{(1+e\cos f)}{(1-e^2)^{1/2}} n\,\delta t_o \right.$$

$$\left. + \left(\frac{2+e\cos f}{1+e\cos f}\right)\sin f\,\delta e + \frac{(1-e^2)}{(1+e\cos f)}\delta\phi\right] \qquad (6\text{-}18)$$

$$\delta z = \frac{h^2}{\mu(1+e\cos f)}\delta i\,(\sin f\cos\delta\Omega - \cos f\sin\delta\Omega) \qquad (6\text{-}19)$$

When (6-17), (6-18), and (6-19) are compared with (4-44), (4-45), and (5-5), the six constants of integration k_1 through k_6 can be written in terms of the orbital element variations.

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \end{bmatrix} = \begin{bmatrix} \delta\phi - \dfrac{n\,\delta t_o}{(1-e^2)^{3/2}} \\ -\dfrac{h^2}{\mu(1-e^2)}\delta e \\ -\dfrac{h^2 e}{\mu(1-e^2)^{3/2}} n\,\delta t_o \\ \dfrac{h^2}{\mu}\dfrac{\delta a}{a} \\ -\delta i\sin\delta\Omega \\ \delta i\cos\delta\Omega \end{bmatrix} \qquad (6\text{-}20)$$

7. The State Vector

The "state" of a space vehicle on its actual trajectory at some specified time t_i may be defined in terms of a six-component vector $\underline{x}_i$ which consists of the three components of position variation and the three components of velocity variation at that time.

$$\underline{x}_i = \begin{bmatrix} \delta \underline{r}_i \\ \delta \dot{\underline{r}}_i \end{bmatrix} \tag{7-1}$$

The subscript i refers to conditions existing at time t_i.

The first three components of $\underline{x}$ at some generalized time t are given in the r s z coordinate system by Equations (6-17), (6-18), and (6-19). The last three components are obtained by vector differentiation. The state vector can then be expressed as the matrix product of a 6-by-6 time-varying matrix and a six-component vector composed of the variations in the orbital elements. The elements of the 6-by-6 matrix can be simplified by the use of the equations for the components of position, velocity, and acceleration along the reference trajectory. The resulting equation is (7-2).

The position components are

$$x = r \cos f = a(\cos E - e) \tag{7-3}$$

$$y = r \sin f = a(1 - e^2)^{1/2} \sin E \tag{7-4}$$

The magnitude of the velocity vector $\dot{\underline{r}}$ is

$$\begin{aligned} v = |\dot{\underline{r}}| &= \left[\mu \left(\frac{2}{r} - \frac{1}{a}\right)\right]^{1/2} \\ &= \frac{\mu}{h} (1 + 2 e \cos f + e^2)^{1/2} \\ &= n a \left(\frac{1 + e \cos E}{1 - e \cos E}\right)^{1/2} \end{aligned} \tag{7-5}$$

The angle between the reference velocity vector $\dot{\underline{r}}$ and the s-axis is the flight path angle γ; the angle between $\dot{\underline{r}}$ and the y-axis is g. The components of $\dot{\underline{r}}$ are

$$v_r = \dot{r} = v \sin \gamma = \frac{\mu}{h} e \sin f = \frac{h e \sin E}{(1 - e^2)^{1/2} r} \tag{7-6}$$

$$\begin{bmatrix} \delta r \\ \delta s \\ \delta z \\ \delta v_r \\ \delta v_s \\ \delta v_z \end{bmatrix} = \begin{bmatrix} r - \frac{3}{2} v_r t & \frac{v_r}{n} & - a \cos f & 0 & 0 & 0 \\ - \frac{3}{2} v_s t & \frac{v_s}{n} & \frac{y}{1 - e^2} + a \sin f & r & 0 & 0 \\ 0 & 0 & 0 & 0 & y & - x \\ - \frac{v_r}{2} - \frac{3}{2} a_r t & \frac{a_r}{n} & - \frac{v_s \sin f}{1 - e^2} & - v_s & 0 & 0 \\ - \frac{v_s}{2} & 0 & \frac{v_y}{1 - e^2} & v_r & 0 & 0 \\ 0 & 0 & 0 & 0 & v_y & - v_x \end{bmatrix} \begin{bmatrix} \frac{\delta a}{a} \\ \delta M_o \\ \delta e \\ \delta \phi \\ \delta i \cos \delta \Omega \\ \delta i \sin \delta \Omega \end{bmatrix} \tag{7-2}$$

$$v_s = r\,\dot{f} = v\cos\gamma = \frac{\mu}{h}\,(1 + e\cos f) = \frac{h}{r} \tag{7-7}$$

$$v_x = \dot{x} = -\,v\sin g = -\frac{\mu}{h}\sin f = -\,\frac{h\sin E}{(1 - e^2)^{1/2}\;r} \tag{7-8}$$

$$v_y = \dot{y} = v\cos g = \frac{\mu}{h}\,(\cos f + e) = \frac{h\cos E}{r} \tag{7-9}$$

The magnitude of the reference acceleration vector $\underline{\ddot{r}}$ is designated a_r, since the vector is in the radial direction.

$$a_r = \left|\,\underline{\ddot{r}}\,\right| = \ddot{r} - r\,\dot{f}^2 = -\,\frac{\mu}{r^2} \tag{7-10}$$

The physical significance of the equations in (7-2) is discussed in Reference (1).

Because the p q z coordinate system is used in the developments contained in the following sections of this paper, the equations for the components of the state vector in that system are presented in (7-11). The relations for the additional components that have been introduced are the following:

$$p = r\cos\gamma = \frac{h}{v} \tag{7-12}$$

$$q = r\sin\gamma = \frac{h}{v}\tan\gamma \tag{7-13}$$

$$\gamma = \tan^{-1}\left[\frac{e\sin f}{1 + e\cos f}\right] = \tan^{-1}\left[\frac{e\sin E}{(1 - e^2)^{1/2}}\right] \tag{7-14}$$

Since the direction of $\underline{\dot{r}}$ is along the q-axis,

$$v_p = 0 \tag{7-15}$$

$$v_q = v \tag{7-16}$$

The acceleration components are

$$a_p = -\,\dot{g}\,v = -\,\frac{\dot{p}(p\dot{p} + q\dot{q})}{q^2} = -\,\mu\,\frac{p}{r^3} \tag{7-17}$$

$$
\begin{bmatrix} \delta p \\ \delta q \\ \delta z \\ \delta v_p \\ \delta v_q \\ \delta v_z \end{bmatrix}
=
\begin{bmatrix}
p & 0 & -a\cos g - \dfrac{y\sin\gamma}{1-e^2} & -q & 0 & 0 \\
q - \dfrac{3}{2}vt & \dfrac{v}{n} & 2a\sin g & p & 0 & 0 \\
0 & 0 & 0 & 0 & y & -x \\
-\dfrac{3}{2}a_p t & \dfrac{a_p}{n} & -\dfrac{v\sin f + v_s\cos f\sin\gamma}{1-e^2} & -v & 0 & 0 \\
-\dfrac{v}{2} - \dfrac{3}{2}a_q t & \dfrac{a_q}{n} & \dfrac{v_s\cos f\cos\gamma}{1-e^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & v_y & -v_x
\end{bmatrix}
\begin{bmatrix} \dfrac{\delta a}{a} \\ \delta M_o \\ \delta e \\ \delta\phi \\ \delta i\cos\delta\Omega \\ \delta i\sin\delta\Omega \end{bmatrix}
\tag{7-11}
$$

$$a_q = \dot{v} = \frac{\dot{p}\dot{p} + q\ddot{q}}{q} + \frac{\dot{p}(\dot{p}q - \dot{q}p)}{q^2} = -\mu \frac{q}{r^3} \tag{7-18}$$

8. The Transition Matrix

The 6-by-6 matrix which relates the state vector $\underline{x}_j$ to the state vector $\underline{x}_i$ is known as the transition matrix $\overset{*}{C}_{ji}$.

$$\underline{x}_j = \overset{*}{C}_{ji}\,\underline{x}_i \tag{8-1}$$

The transition matrix has some interesting properties which are valid for many-body gravitational fields as well as two-body fields. In the first place, it is obvious that, with the subscript notation being used,

$$\overset{*}{C}_{ii} = \overset{*}{I}_6 \tag{8-2}$$

where $\overset{*}{I}_6$ is the 6-by-6 identity matrix. Secondly, the inverse of the transition matrix is given by

$$\overset{*}{C}_{ji}{}^{-1} = \overset{*}{C}_{ij} \tag{8-3}$$

where the superscript -1 signifies the inverse. The third property also has to do with the inverse. $\overset{*}{C}_{ji}$ can be expressed in terms of four 3-by-3 submatrices.

$$\overset{*}{C}_{ji} = \begin{bmatrix} \overset{*}{M}_{ji} & \overset{*}{N}_{ji} \\ \overset{*}{S}_{ji} & \overset{*}{T}_{ji} \end{bmatrix} \tag{8-4}$$

Then the inverse can be shown to be

$$\overset{*}{C}_{ji}{}^{-1} = \begin{bmatrix} \overset{*}{T}_{ji}{}^{T} & -\overset{*}{N}_{ji}{}^{T} \\ -\overset{*}{S}_{ji}{}^{T} & \overset{*}{M}_{ji}{}^{T} \end{bmatrix} \tag{8-5}$$

This relationship is very useful, because it allows the 6-by-6 transition matrix to be inverted by inspection. Its utilization is explained in greater detail in Reference (4). The fourth property is that, despite the fact that the non-zero elements of the transition matrix vary with both t_i and t_j, its determinant is always equal to unity.

$$\left|\overset{*}{C}_{ji}\right| = +1 \tag{8-6}$$

This equation is helpful in checking the accuracy of numerical computations of the elements of $\overset{*}{C}_{ji}$.

For elliptical reference trajectories, the results of Section 7 can be used to determine analytic expressions for the elements of $\overset{*}{C}_{ji}$. If a six-component vector $\underline{e}$ is defined as some linearly independent combination of the variations in the orbital elements and if the components of the state vector $\underline{x}$ are rearranged to form a modified state vector $\underline{x}'$ whose four in-plane components precede the two normal components, then $\underline{x}_j'$ is related to $\underline{e}$ by means of the 6-by-6 matrix $\overset{*}{Y}_j$.

$$\underline{x}_j' = \overset{*}{Y}_j \, \underline{e} \tag{8-7}$$

Since $\overset{*}{Y}_j$ is non-singular for all values of time,

$$\underline{e} = \overset{*}{Y}_j^{-1} \, \underline{x}_j' \tag{8-8}$$

Then,

$$\underline{x}_j' = \overset{*}{Y}_j \, \overset{*}{Y}_i^{-1} \, \underline{x}_i' \tag{8-9}$$

and a modified transition matrix $\overset{*}{C}_{ji}'$ can be expressed as the product of two matrices, one of which varies only with t_j and the other varies only with t_i.

$$\overset{*}{C}_{ji}' = \overset{*}{Y}_j \, \overset{*}{Y}_i^{-1} \tag{8-10}$$

The actual transition matrix C_{ji}, defined by Equations (7-1) and (8-1), is obtained from $\overset{*}{C}_{ji}'$ by a rearrangement of matrix elements.

The amount of algebraic and trigonometric manipulation involved in performing the matrix multiplication of (8-10) is quite formidable. It is therefore prudent to choose with great care the coordinate system in which $\underline{x}$ and $\underline{x}'$ are expressed and the orbital elements which comprise $\underline{e}$. The p q z coordinate system has been found to be most appropriate for expressing $\underline{x}$ and $\underline{x}'$; in this system $\underline{x}$ is given by the column vector on the left side of Equation (7-11), and the order of the components of $\underline{x}'$ is the following:

$$\delta p, \quad \delta q, \quad \delta v_p, \quad \delta v_q, \quad \delta z, \quad \delta v_z.$$

The p q z coordinate system is used exclusively in the analytic expressions in the remainder of this paper. The components chosen for $\underline{e}$ are those shown in Equation (8-11).

$$
\underline{e} = \begin{Bmatrix} (1 - e^2)^{1/2} \ \delta\phi - n\ \delta t_o \\ \dfrac{\delta e}{(1 - e^2)^{1/2}} \\ \dfrac{1}{2} \ \dfrac{\delta a}{a} \\ e\ \delta\phi \\ (1 - e^2)^{1/2} \ \delta i \cos \delta\Omega \\ \delta i \sin \delta\Omega \end{Bmatrix} \tag{8-11}
$$

As a further aid in simplifying the algebra, all time-varying quantities are expressed in terms of the eccentric anomaly.

With these selections the elements of $\overset{*}{Y}$ and its inverse are given by Equations (8-12) and (8-13). The dashed lines in the equations indicate matrix partitioning. It is interesting to compare the submatrix composed of the first four rows and the first four columns of $\overset{*}{Y}$ with the corresponding submatrix of $\overset{*}{Y}^{-1}$. Let the former be designated $\overset{*}{Y}_4$, and let it be further divided into four 2-by-2 submatrices.

$$
\overset{*}{Y}_4 = \begin{bmatrix} \overset{*}{A}_1 & \overset{*}{A}_2 \\ \overset{*}{A}_3 & \overset{*}{A}_4 \end{bmatrix} \tag{8-14}
$$

Examination of (8-12) and (8-13) shows that $\overset{*}{Y}_4{}^{-1}$ is given by

$$
\overset{*}{Y}_4{}^{-1} = \frac{(1 - e^2)^{1/2}}{h} \begin{bmatrix} \overset{*}{A}_4{}^T & -\overset{*}{A}_2{}^T \\ -\overset{*}{A}_3{}^T & \overset{*}{A}_1{}^T \end{bmatrix} \tag{8-15}
$$

$$
Y = a \left\{ \begin{array}{c}
\dfrac{1}{(1-e^2\cos^2 E)^{1/2}} \left[\begin{array}{l}
(1-e\cos E) \begin{pmatrix} 0 & -(\cos E+e) & 2(1-e^2)^{1/2} & -\sin E & 0 & 0 \end{pmatrix} \\
\hline
\begin{pmatrix} 1+e\cos E & 2(1-e^2)^{1/2}\sin E & \begin{array}{l} -3(E-e\sin E)(1+e\cos E) \\ +2e\sin E(1-e\cos E) \end{array} & -2(1-e^2)^{1/2}\cos E & 0 & 0 \end{pmatrix} \\
\hline
\dfrac{n}{(1-e\cos E)^2} \begin{pmatrix} -(1-e^2)^{1/2} & -\left[\begin{array}{l} (1-e\cos E)e\cos E \\ +(1-e^2) \end{array} \right]\sin E & 3(E-e\sin E)(1-e^2)^{1/2} & \begin{array}{l} (1-e\cos E)e\cos^2 E \\ +(\cos E-e) \end{array} & 0 & 0 \\ -e\sin E & (1-e^2)^{1/2}(\cos E-e) & \begin{array}{l} 3(E-e\sin E)e\sin E \\ -(1+e\cos E)(1-e\cos E)^2 \end{array} & (1-e^2)^{1/2}\sin E & 0 & 0 \end{pmatrix}
\end{array} \right] \\
\hline
\begin{pmatrix} 0 & 0 & 0 & 0 & \sin E & -(\cos E-e) \end{pmatrix} \\
\hline
\dfrac{n}{1-e\cos E} \begin{pmatrix} 0 & 0 & 0 & 0 & \cos F & \sin E \end{pmatrix}
\end{array} \right\}
$$

(8-12)

$$
\overset{*}{Y}{}^{-1} = \frac{1}{a} \left\{ \frac{1}{(1-e^2\cos^2 E)^{1/2}} \left[\frac{1}{(1-e\cos E)^2} \begin{pmatrix} 3(E-e\sin E)(1-e^2)^{1/2} & \begin{array}{l} 3(E-e\sin E)e\sin E \\ -(1+e\cos E)(1-e\cos E)^2 \end{array} \\ \begin{array}{l} (1-e\cos E)e\cos^2 E \\ +(\cos E-e) \end{array} & (1-e^2)^{1/2}\sin E \\ (1-e^2)^{1/2} & e\sin E \\ \left[\begin{array}{l} (1-e\cos E)e\cos E \\ +(1-e^2) \end{array} \right]\sin E & -(1-e^2)^{1/2}(\cos E-e) \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \middle| \frac{1-e\cos E}{n} \begin{pmatrix} -2(1-e^2)^{1/2} \\ \sin E \\ 0 \\ -(\cos E+e) \\ 0 \\ 0 \end{pmatrix} \middle| \frac{1}{n} \begin{pmatrix} \begin{array}{l} 3(E-e\sin E)(1+e\cos E) \\ -2e\sin E(1-e\cos E) \end{array} \\ 2(1-e^2)^{1/2}\cos E \\ 1+e\cos E \\ 2(1-e^2)^{1/2}\sin E \\ 0 \\ 0 \end{pmatrix} \right] \middle| \frac{1}{1-e\cos E} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \sin E \\ -\cos E \end{pmatrix} \middle| \frac{1}{n} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cos F - e \\ \sin E \end{pmatrix} \right\}
$$

(8-13)

$$
\left\{
\begin{bmatrix}
\left[\frac{(1+e\cos E_i)(1-e\cos E_j)}{(1-e\cos E_i)(1+e\cos E_j)}\right]^{1/2} & 0 & 0 & 0 & 0 & 0 \\
0 & \left[\frac{(1-e\cos E_i)(1+e\cos E_j)}{(1+e\cos E_i)(1-e\cos E_j)}\right]^{1/2} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \left[\frac{(1-e\cos E_i)(1+e\cos E_j)}{(1+e\cos E_i)(1-e\cos E_j)}\right]^{1/2} & 0 & 0 \\
0 & 0 & 0 & 0 & \left[\frac{(1+e\cos E_i)(1-e\cos E_j)}{(1-e\cos E_i)(1+e\cos E_j)}\right]^{1/2} & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\right\}
+ 2
\left\{
\begin{array}{l|l}
\frac{1}{(1-e^2\cos^2 E_i)^{1/2}(1-e^2\cos^2 E_j)^{1/2}} & \left[\frac{1}{(1-e\cos E_i)^2} \left(\begin{array}{ll}
(1-e\cos E_i)\left[(1-e^2)\sin E_M \right. & (1-e^2)^{1/2}(1-e\cos E_j) \\
\left. -(1-e\cos E_i)\,e\sin E_i\cos E_M\right]\sin E_M & \cdot\sin E_M(\cos E_M - e\cos E_P) \\
 & \\
(1-e^2)^{1/2}\left\{-(1+e\cos E_j)\right. & -(1-e\cos E_j)\,e\sin E_i \\
\cdot(3E_M - e\sin E_M\cos E_P) & (3E_M - e\sin E_M\cos E_P) \\
+2\sin E_M\left[e\cos E_P\right. & +2\sin E_M\left[2e\sin E_P\right. \\
\left.\left.-(1+e\cos E_i - e^2\cos^2 E_i)\cos E_M\right]\right\} & \left.-(1+e^2)\sin E_M\right]
\end{array} \right. \right. \\
\hline
\sin E_M & \left[\begin{array}{ll} \qquad 0 & \qquad 0 \end{array} \right. \\
\hline
\frac{1}{(1-e^2\cos^2 E_i)^{1/2}(1-e^2\cos^2 E_j)^{1/2}(1-e\cos E_j)^2} & \left[\frac{n}{(1-e\cos E_i)^2} \left(\begin{array}{ll}
(1-e^2)(3E_M - 2e\sin E_M\cos E_P) & (1-e^2)^{1/2}\left\{3e\sin E_i(E_M - e\sin E_M\cos E_P)\right. \\
-\left\{(1-e\cos E_i)(1-e\cos E_j)e^2\sin E_i\sin E_j\right. & +\left[(1-e^2)+(1-e\cos E_j)e\cos E_j\right]\sin^2 E_M \\
+(1-e^2)\left[1+(1-e\cos E_i)e\cos E_i\right. & -e^2\left[2(1-e\cos E_j)\cos E_M\cos E_P\right. \\
\left.\left.+(1-e\cos E_j)e\cos E_j\right]\right\}\sin E_M\cos E_M & \left.\left.+(1-e\cos E_j)\cos E_j\right]\sin E_M\sin E_P\right\} \\
 & \\
(1-e^2)^{1/2}\left\{3e\sin E_j(E_M - e\sin E_M\cos E_P)\right. & e^2\sin E_i\sin E_j\left[3E_M - 4e\sin E_M\cos E_P\right. \\
-\left[(1-e^2)+(1-e\cos E_i)e\cos E_i\right]\sin^2 E_M & \left.+e^2\sin E_M\cos E_M(\cos^2 E_P - \sin^2 E_P)\right] \\
-e^2\left[2(1-e\cos E_j)\cos E_M\cos E_P\right. & -(1-e^2)\sin E_M\left[(1+e^2)\cos E_M - 2e\cos E_P\right] \\
\left.\left.+(1-e\cos E_i)\cos E_i\right]\sin E_M\sin E_P\right\} &
\end{array} \right. \right. \\
\hline
-\frac{\sin E_M}{1-e\cos E_j} & \left[\begin{array}{ll} \qquad 0 & \qquad 0 \end{array} \right.
\end{array}
\right.
$$

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$$
\left.\begin{array}{c|c|c|c}
0 & \frac{1}{n}\,(1 - e\cos E_j)(1 - e\cos E_j)\cdot\sin E_M(\cos E_M + e\cos E_P) & \frac{1}{n}\,2(1-e^2)^{1/2}(1 - e\cos E_j)\sin^2 E_M & 0 \\
0 & -\frac{1}{n}\,2(1-e^2)^{1/2}(1 - e\cos E_j)\sin^2 E_M & \frac{1}{n}\left[-(1 + e\cos E_j)(1 + e\cos E_j)\cdot(3E_M - e\sin E_M\cos E_P) + 4\sin E_M(\cos E_M + e\cos E_P)\right] & 0 \\
\hline
-\dfrac{\sin E_M}{1 - e\cos E_j} & 0 & 0 & \frac{1}{n}(\cos E_M - e\cos E_P) \\
\hline
0 & \frac{1}{n}\,(1 - e\cos E_j)\left[(1-e^2)\sin E_M + (1 - e\cos E_j)\,e\sin E_j\cos E_M\right]\sin E_M & \frac{1}{n}\,(1-e^2)^{1/2}\left\{(1 + e\cos E_j)\cdot(3E_M - e\sin E_M\cos E_P) - 2\sin E_M\left[e\cos E_P + (1 + e\cos E_j - e^2\cos^2 E_j)\cos E_M\right]\right\} & 0 \\
0 & -\frac{1}{n}\,(1-e^2)^{1/2}(1 - e\cos E_j)\cdot\sin E_M(\cos E_M - e\cos E_P) & \frac{1}{n}\left\{(1 + e\cos E_j)\,e\sin E_j\cdot(3E_M - e\sin E_M\cos E_P) - 3\sin E_M\left[2e\sin E_P + (1+e^2)\sin E_M\right]\right\} & 0 \\
\hline
\dfrac{n\cos E_M}{1 - e\cos E_j} & 0 & 0 & \sin E_M
\end{array}\right\}
\qquad (8\text{-}16)
$$

This characteristic, which is due to the particular choice made for $\underline{e}$, is helpful in the determination of the elements of $\overset{*}{C}_{ji}$.

The final equation for $\overset{*}{C}_{ji}$ for elliptical reference trajectories is (8-16). The angles E_M and E_P, introduced in this equation, are defined as follows:

$$E_M = \frac{1}{2} (E_j - E_i) \tag{8-17}$$

$$E_P = \frac{1}{2} (E_j + E_i) \tag{8-18}$$

Equation (8-16) expresses all the elements of $\overset{*}{C}_{ji}$ in terms of only four quantities, namely the eccentricity e and the mean angular motion n of the reference trajectory, and the eccentric anomalies E_i and E_j. $\overset{*}{C}_{ji}$ is presented as the sum of two matrices. The first of the two is a diagonal matrix whose diagonal elements, reading from top to bottom, are equal, respectively, to

$$\frac{v_i}{v_j} \quad , \quad \frac{v_j}{v_i} \quad , \quad 1 \quad , \quad \frac{v_j}{v_i} \quad , \quad \frac{v_i}{v_j} \quad , \quad 1 \quad .$$

Every term in every element of the second of the two matrices contains either E_M or sin E_M. From this information it is clear that when $t_j = t_i$, the first matrix becomes the identity matrix and the second matrix becomes the zero matrix. Thus, $\overset{*}{C}_{ii}$ is the sixth-order identity matrix, as indicated by Equation (8-2).

For this special case, in which the reference trajectory is an ellipse and one of the reference trajectory coordinate systems is used, a stronger statement can be made about the determinant of $\overset{*}{C}_{ji}$ than that of Equation (8-6). Not only is the determinant of $\overset{*}{C}_{ji}$ equal to unity, but there are two sub-matrices that can be formed, the determinant of each of which is equal to unity. The first sub-matrix consists of the sixteen in-plane elements of $\overset{*}{C}_{ji}$; the second consists of the four out-of-plane elements of $\overset{*}{C}_{ji}$.

9. The Correction Matrix

In addition to the vectors $\underline{x}$ and $\underline{e}$, another useful set of constants defining the vehicle's variant path consists of the two position variations $\delta \underline{r}_i$ and $\delta \underline{r}_j$.

In particular, it is desirable to be able to express the velocity variations $\delta \underline{v}_i$ and $\delta \underline{v}_j$ in terms of $\delta \underline{r}_i$ and $\delta \underline{r}_j$. The equations have the form

$$\delta \underline{v}_i = \overset{*}{J}_{ij}\, \delta \underline{r}_i + \overset{*}{K}_{ij}\, \delta \underline{r}_j \tag{9-1}$$

$$\delta \underline{v}_j = \overset{*}{K}_{ji}\, \delta \underline{r}_i + \overset{*}{J}_{ji}\, \delta \underline{r}_j \tag{9-2}$$

By pre-multiplying both sides of (9-1) by $\overset{*}{K}_{ij}{}^{-1}$, $\delta \underline{r}_j$ can be written in terms of $\underline{x}_i$.

$$\delta \underline{r}_j = \begin{bmatrix} -\overset{*}{K}_{ij}{}^{-1}\, \overset{*}{J}_{ij} & \overset{*}{K}_{ij}{}^{-1} \end{bmatrix} \underline{x}_i \tag{9-3}$$

From (8-1) and (8-4),

$$\delta \underline{r}_j = \begin{bmatrix} \overset{*}{M}_{ji} & \overset{*}{N}_{ji} \end{bmatrix} \underline{x}_i \tag{9-4}$$

Since $\underline{x}_i$ is an arbitrary vector,

$$\overset{*}{M}_{ji} = -\overset{*}{K}_{ij}{}^{-1}\, \overset{*}{J}_{ij} \tag{9-5}$$

$$\overset{*}{N}_{ji} = \overset{*}{K}_{ij}{}^{-1} \tag{9-6}$$

The last two equations can be solved for $\overset{*}{J}_{ij}$ and $\overset{*}{K}_{ij}$ in terms of the submatrices of $\overset{*}{C}_{ji}$ or $\overset{*}{C}_{ij}$.

$$\overset{*}{J}_{ij} = -\overset{*}{N}_{ji}{}^{-1}\, \overset{*}{M}_{ji} = (\overset{*}{N}_{ij}{}^{T})^{-1}\, \overset{*}{T}_{ij}{}^{T} \tag{9-7}$$

$$\overset{*}{K}_{ij} = \overset{*}{N}_{ji}{}^{-1} = -(N_{ij}{}^{T})^{-1} \tag{9-8}$$

It is shown on Page 697 of Reference (5) and in Appendix F of Reference (1) that $\overset{*}{J}_{ij}$ is a symmetrical matrix.

The preceding material describes the variant motion of a space vehicle in a gravitational field. Once the variant motion is known, the problem is to alter this motion by means of a midcourse correction so that the objective of the mission can be achieved.

The means of altering the motion is a short application of thrust from a reaction-type engine. Because the thrust application is so short in duration relative to the time required for the space voyage, it is treated mathematically as a thrust impulse. At the time of the correction, t_C, there is a step change in vehicle velocity but no instantaneous change in vehicle position.

Obviously, the correction causes a change in the state vector $\underline{x}$ which characterizes the variant motion. The superscripts - and + will be used to distinguish conditions applicable before the correction from those applicable after the correction. The change in the state vector $\underline{x}_C$ is given by

$$\underline{x}_C^{+} - \underline{x}_C^{-} = \begin{bmatrix} \underline{0}_3 \\ \underline{c} \end{bmatrix} \tag{9-9}$$

where $\underline{0}_3$ is the three-dimensional zero vector and $\underline{c}$ is the velocity correction vector. The three components of $\underline{c}$ are to be computed in such a manner that three specified design conditions are satisfied.

It is apparent that a single correction cannot cause the vehicle to return immediately to its reference trajectory, because accomplishing this would require that six conditions be met (i.e., $\delta \underline{r} = \underline{0}_3$, $\delta \underline{v} = \underline{0}_3$). The three design conditions that are to be satisfied are generally associated with the vehicle's state vector $\underline{x}_D$ at the time of its arrival at the destination. Thus the correction is intended to establish a new variant path which modifies $\underline{x}_D$ in some desired fashion. The difference between the corrected and the original state vectors at time t_D is related to the corresponding difference in state vectors at t_C by the equation

$$\begin{aligned} \underline{x}_D^{+} - \underline{x}_D^{-} &= \overset{*}{C}_{DC} (\underline{x}_C^{+} - \underline{x}_C^{-}) \\ &= \overset{*}{C}_{DC} \begin{Bmatrix} \underline{0}_3 \\ \underline{c} \end{Bmatrix} = \begin{Bmatrix} \overset{*}{N}_{DC} \\ \overset{*}{T}_{DC} \end{Bmatrix} \underline{c} \end{aligned} \tag{9-10}$$

The only type of guidance to be considered in this paper is fixed-time-of-arrival guidance, in which it is stipulated that the space vehicle arrive at the destination at the exact time specified by the reference trajectory. Thus, the three mathematical conditions to be satisfied by the midcourse correction are contained in the simple equation

$$\delta \underline{r}_D^{\,+} = \underline{0}_3 \tag{9-11}$$

Equations (9-10) and (9-11) can be solved for the correction $\underline{c}$.

$$\begin{aligned} \underline{c} &= \overset{*}{N}_{DC}^{\,-1} \left(\delta \underline{r}_D^{\,+} - \delta \underline{r}_D^{\,-}\right) \\ &= - \overset{*}{K}_{CD}\, \delta \underline{r}_D^{\,-} \end{aligned} \tag{9-12}$$

Thus, despite the fact that six quantities are required to define the variant path completely, only three (the components of $\delta \underline{r}_D^{\,-}$) are needed to determine the fixed-time-of-arrival velocity correction. Matrix $\overset{*}{K}_{CD}$ is designated the correction matrix.

The resulting velocity variation $\delta \underline{v}_D^{\,+}$ at the destination can be determined from (9-10), with the aid of (9-7) and (9-8).

$$\begin{aligned} \delta \underline{v}_D^{\,+} &= \delta \underline{v}_D^{\,-} + \overset{*}{T}_{DC}\, \underline{c} \\ &= \delta \underline{v}_D^{\,-} - \overset{*}{T}_{DC}\, \overset{*}{K}_{CD}\, \delta \underline{r}_D^{\,-} \\ &= \left[- \overset{*}{J}_{DC} \qquad \overset{*}{I}_3 \right] \underline{x}_D^{\,-} \end{aligned} \tag{9-13}$$

The new and the old state vectors at t_D are related by the equation

$$\underline{x}_D^{\,+} = \begin{bmatrix} \overset{*}{0}_3 & \overset{*}{0}_3 \\ -\overset{*}{J}_{DC} & \overset{*}{I}_3 \end{bmatrix} \underline{x}_D^{\,-} \tag{9-14}$$

Equations (9-12) and (9-14) are applicable to many-body gravitational fields. When the reference trajectory is an ellipse, analytic expressions for the elements of the required matrices $\overset{*}{K}_{CD}$ and $\overset{*}{J}_{DC}$ can be found from Equations (9-7), (9-8), and the relevant submatrices appearing in Equation (8-16). These expressions are given in Equations (9-15) and (9-16). In (9-15), because subscript C precedes subscript D in $\overset{*}{K}_{CD}$,

$$E_M = \frac{1}{2} \left(E_D - E_C\right) \tag{9-17}$$

$$\overset{\ddagger}{K}_{CD} = \frac{n}{2} \left\{ \begin{array}{c} \dfrac{1}{(1 - e^2 \cos^2 E_C)^{1/2} (1 - e^2 \cos^2 E_D)^{1/2}} \times \begin{pmatrix} \begin{array}{l} (1 + e \cos E_C)(1 + e \cos E_D) \\ \cdot \left(\dfrac{3 E_M}{\sin E_M} - e \cos E_P \right) \\ - 4(\cos E_M + e \cos E_P) \end{array} & 2(1 - e^2)^{1/2} (1 - e \cos E_D) \sin E_M & 0 \\ - 2(1 - e^2)^{1/2} (1 - e \cos E_C) \sin E_M & \begin{array}{l} -(1 - e \cos E_C)(1 - e \cos E_D) \\ \cdot (\cos E_M + e \cos E_P) \end{array} & 0 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & 0 & \dfrac{1}{\sin E_M (\cos E_M - e \cos E_P)} \end{pmatrix} \end{array} \right\} \tag{9-15}$$

$$
\overset{*}{J}_{DC} = n \left\{ \begin{array}{c}
\dfrac{1}{1 + e\cos E_D} \begin{pmatrix}
\begin{array}{l}
\dfrac{e \sin E_D}{(1 - e\cos E_D)^2} \\
+ \dfrac{1}{2(1 - e\cos E_D)X} \Big\{ (1 + e\cos E_D) \\
\cdot \left[2\sin^2 E_M - (1 + e\cos E_D) \right] \\
\cdot \left[\dfrac{3 E_M}{\sin E_M} - e\cos E_P \right] \\
+ 4 \Big[(1 + e\cos E_D)\cos E_M \\
+ e \sin E_D \sin E_M \Big] \Big\}
\end{array} & -(1 - e^2)^{1/2} \left[\dfrac{1}{(1 - e\cos E_D)^2} + \dfrac{\sin E_M}{X} \right] & 0 \\
-(1 - e^2)^{1/2} \left[\dfrac{1}{(1 - e\cos E_D)^2} + \dfrac{\sin E_M}{X} \right] & \begin{array}{l}
- \dfrac{e \sin E_D}{(1 - e\cos E_D)^2} \\
+ \dfrac{1}{2X}(1 - e\cos E_D)(\cos E_M + e\cos E_P)
\end{array} & 0
\end{pmatrix} \\
\hdashline
\begin{pmatrix} 0 & 0 & \dfrac{2\sin^2 E_M - (1 - e\cos E_D)}{2\sin E_M(\cos E_M - e\cos E_P)(1 - e\cos E_D)} \end{pmatrix}
\end{array} \right\}
$$

(9-16)

On the other hand, subscript D precedes subscript C in $\overset{*}{J}_{DC}$, and therefore in (9-16)

$$E_M = \frac{1}{2}(E_C - E_D) \tag{9-18}$$

The denominator factor X in the expressions for $\overset{*}{J}_{DC}$ and $\overset{*}{K}_{CD}$ is defined by

$$X = (3\,E_M - e \sin E_M \cos E_P)(\cos E_M + e \cos E_P) - 4 \sin E_M \tag{9-19}$$

The matrix $\overset{*}{N}_{DC}$ (or $\overset{*}{N}_{CD}$), which must be inverted in order to obtain $\overset{*}{J}_{DC}$ and $\overset{*}{K}_{CD}$, becomes singular whenever the value of the true anomaly difference $(f_D - f_C)$ is equal to an integer multiple of π radians and also when the factor X becomes zero. The smallest positive value of $(f_D - f_C)$ for which X goes to zero is always greater than 2π radians. Therefore, for manned interplanetary flights, in which the total transfer angle is normally not greater than π radians, there is little cause for alarm about the singularities. A detailed discussion of the singularities is contained in Appendix O of Reference (1).

10. Comments on the Application of the Analytic Solution

The analytic solution of the linearized variant equations for elliptical motion provides a relatively simple and yet quite accurate means of utilizing impulsive thrust for the midcourse guidance of an interplanetary space vehicle. Once the state vector $\underline{x}_D$ has been estimated from measurements made during the flight, the velocity correction $\underline{c}$ can be directly determined from Equations (9-12) and (9-15). There are only five non-zero elements in the required $\overset{*}{K}_{CD}$ matrix; these can readily be determined as functions of E_C, the eccentric anomaly at the time of the correction.

The accuracy of the computation is enhanced by the fact that there is no build-up of round-off error, as would be the case if numerical integration were required. In the present solution all integration has been performed analytically, before the numerical computations are started.

In similar fashion, there is a straightforward determination of the transition matrix $\overset{*}{C}_{ji}$, which is useful in predicting $\underline{x}_D$, and of the matrix $\overset{*}{J}_{DC}$, which is used in obtaining the change in $\underline{x}_D$ due to the correction.

The two-body assumption requires that the nominal destination point be appreciably beyond the sphere of influence of the destination planet if high accuracy is to be maintained. Although no numerical studies have yet been made, a radial distance from the planet of the order of one million miles is felt to provide a conservative destination point for trips to Venus or Mars. The distant destination point is not a severe limitation on the applicability of the analytic solution, since in most missions some type of terminal guidance is required during the final phase of the journey.

Although the analysis of Section 6 makes use of the assumption that the reference trajectory is an ellipse of moderate eccentricity, the final equations for $\overset{*}{C}_{ji}$, $\overset{*}{J}_{DC}$, and $\overset{*}{K}_{CD}$ are also applicable to circular reference trajectories; they are not applicable if the eccentricity of the reference trajectory appraoches very close to unity. It must be kept in mind that the analytic expressions given for these matrices are based on the p q z coordinate system.

The computation of the correction from a knowledge of the estimated position variation at the destination can readily be accomplished by a small-scale digital computer of the type that would normally be carried on board the space vehicle. For this reason it is possible to leave some latitude in the final choice of the time at which the correction is to be applied. If it is found to be desirable, the correction time can be made a function of the state vector $\underline{x}_D$ inferred from the measurements made during the flight.

On manned interplanetary flights it is possible for the astronauts to compute manually the velocity correction corresponding to an estimated position variation and a specified time of correction. The manual computation serves as a check of the operation of the digital machine. It may also have some advantages from the psychological viewpoint for trips of long duration.

Finally, the analytic solution can be valuable in guidance along abort trajectories, for which elaborate pre-computation may not be feasible.

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